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Families of IIB duals for nonrelativistic CFTs

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Abstract: We show that the recent string theory embedding of a spacetime with nonrelativistic Schrödinger symmetry can be generalised to a twenty one dimensional family of solutions with that symmetry. Our solutions include IIB backgrounds with no three form flux turned on, and arise as near horizon limits of branewave spacetimes. We show that there is a hypersurface in the space of these theories where an instability appears in the gravitational description, indicating a phase transition in the nonrelativistic field theory dual. We also present simple embeddings of duals for nonrelativistic critical points where the dynamical critical exponent can take many values $z \neq 2$.

Keywords: AdS-CFT Correspondence, Classical Theories of Gravity, Space-Time Symmetries.

Contents

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1. Background

In recent studies of the (strongly coupled) quark-gluon plasma, just above the QCD deconfinement temperature, the AdS/CFT correspondence [1] is finally starting to deliver qualitative and even quantitative results of experimental relevance that are difficult to obtain by more traditional theoretical methods (for overviews see for instance [2 – 4]).

The best understood versions of the AdS/CFT correspondence describe a duality between a conformally invariant gauge theory and string theory in an asymptotically Anti-de Sitter space background. Taking a large N and (if necessary) strong coupling limit in the gauge theory corresponds to taking the Anti-de Sitter space to be weakly curved, and hence string theory may be described by a ten or eleven dimensional supergravity theory.

QCD is a gauge theory, and so it is natural to hope that certain quantities will be in the same 'universality class' as the supersymmetric gauge theories that can be studied using AdS/CFT. This hope seems to be bourn out for the high temperature deconfined phase of QCD [2 − 4]. On the other hand, there are many quantities of interest that are sensitive to the fact that QCD is not a conformal theory, but rather asymptotically free and confining. Extensions of the AdS/CFT correspondence to non-conformal theories are well studied. However, they generically suffer from the feature that parametrically decoupling the confining physics from other scales in the string background (such as the Kaluza-Klein scale) requires going beyond the supergravity approximation. This problem is reviewed for instance in [5].

One is then led to wonder whether there are exactly conformally (or at least, scale) invariant systems in nature that would be kinematically closer to the simplest versions of AdS/CFT. Conformal invariance has in fact been observed to arise in an increasing number of condensed matter systems - in the vicinity of a quantum critical point [6 − 8].

The microscopic description of these condensed matter systems is certainly not relativistic. It is remarkable therefore that certain quantum critical points exhibit an emergence of the full relativistic conformal symmetry. This occurs for instance in certain superfluid-insulator transitions and at certain 'deconfined' quantum critical points (see for instance [9, 10] for references). If a nearby relativistic quantum critical point is believed to play a role in some dynamical process, then one can hope that standard AdS/CFT techniques can be applied. This was the methodological assumption in [10 − 12], which studied certain transport processes in unconventional superconductors.

It is also common for quantum critical points to display a non-relativistic scale invariance. These theories have a scale invariance that treats space and time anisotropically, as captured by the dynamical critical exponent z. Thus for instance, as the critical point is approached, the mass gap Δ, which determines relaxation timescales, and the coherence length ξ are related by

$$\Delta \sim \frac{1}{\xi^z} \,. \tag{1.1}$$

The case $z = 1$ corresponds to a relativistic theory. For general z the symmetry of the theory is simply the Galilean group enhanced by a dilatation operation. The case $z = 2$ is also singled out, as the algebra may be enhanced in this case by an additional 'special conformal' generator. This is called the Schrödinger algebra and enforces strong constraints on the dynamics. We will not review this algebra in detail, for recent discussions see for instance [13 − 15]. Early discussion of the symmetry of the Schrödinger equation are [16, 17] and an early geometric discussion of these symmetries appears in [18]. Examples of experimental systems with $z = 2$ include fermions at unitarity, see [14, 15] for references.

Recent work has suggested that the AdS/CFT correspondence can be adapted to study strongly coupled non-relativistic quantum critical points. We will concentrate in this paper on the most constrained case of $z = 2$, and on field theory in $2 + 1$ dimensions. We shall also present some interesting extensions of our results to general z. The starting point is to write down a metric upon which the Schrödinger symmetry is geometrically realised. The following five dimensional metric does the job [14, 15]

$$ds_5^2 = R^2 \left(r^2 \left(-2dx^+ dx^- + d\boldsymbol{x}^2 \right) + \frac{dr^2}{r^2} - \sigma^2 r^4 (dx^+)^2 \right) \,. \tag{1.2}$$

Here σ is a parameter measuring deformation of the metric away from Anti-de Sitter space. Furthermore, one should periodically identify

$$x^- \sim x^- + 2\pi r^- \,. \tag{1.3}$$

This allows $P_- \equiv i\partial_{x^-}$ to be interpreted as particle number

$$P_- = \frac{N}{r^-}\,, \tag{1.4}$$

with N an integer. If we then consider x^+ to be time, so that $P_+ \equiv i\partial_{x^+}$ is energy, then positivity of energy for classical fields in the spacetime (1.2) requires $N \geq 0$.

The works [14, 15] showed that by generalising standard AdS/CFT arguments to fields propagating in the background metric (1.2) one obtains sensible results for correlators of operators in a Schrödinger invariant theory. In order to exploit the full power of the AdS/CFT correspondence, and obtain, for instance, the finite temperature and number density backgrounds, it is necessary to embed the metric (1.2) into a consistent theory of gravity, such as string or M theory. This was achieved in the works [19 − 21] who obtained the following solution of IIB string theory on $AdS_5 \times X_5$, with X_5 a Sasaki-Einstein space

$$\begin{aligned} ds^2_{\text{IIB}} &= R^2 \left(r^2 \left(-2dx^+dx^- + d\boldsymbol{x}^2\right) + \frac{dr^2}{r^2} - \sigma^2 r^4 (dx^+)^2 + ds^2_{K-E} + \eta^2 \right) \,, && (1.5)\\ B_2 &= \sigma e^{\Phi/2} R^2 r^2 dx^+ \wedge \eta \,, && (1.6)\\ F_5 &= 4R^4(1+\star)\text{vol}(X_5)\,, && (1.7) \end{aligned}$$

where Φ is the dilaton, which is constant. In this expression the Sasaki-Einstein space has been written as a fibration over a local Kähler-Einstein manifold

$$ds^2_{X_5} = ds^2_{K-E} + \eta^2 \,, \tag{1.8}$$

such that $d\eta = 2\omega$, the Kähler form on the Kähler-Einstein base. In the maximally symmetric case $X_5 = S^5$, the Kähler-Einstein manifold is $\mathbb{CP}^2$. Closely related solutions had been found previously in [22, 23].

Before moving on, two comments are in order. Firstly, it was emphasised in [20] that the circle that we have periodically identified in (1.3) is null and therefore has zero proper length. This means that strings winding the circle are light and strictly speaking one cannot trust the supergravity regime. It was also noted how this situation can be ameliorated by considering finite density, as is indeed appropriate for comparison to experimental setups. Secondly, an alternative proposal for realising a gravity dual of non-relativistic theories was presented in [24, 25]. These papers use pure AdS, with the relativistic symmetry being broken by boundary conditions. This approach has been critiqued, in our view correctly, in [20, 19].

In this work we shall present some new solutions to IIB supergravity with a Schrödinger symmetry. There will be a large family of solutions (21 dimensional in the case of $AdS_5 \times S^5$) that interpolate from the solution (1.5) to a solution which only involves the metric. As we move along the space of solutions, an instability develops on a critical hypersurface in parameter space. This instability is presumably dual to a (quantum) phase transition in the coupling space of the quantum field theories.

The single step in our construction of these solutions is to start with $AdS_5 \times X_5$ and then to introduce a harmonic function on X_5 into the AdS_5 part of the metric. An intriguing feature of this construction is that the resulting exponent z is directly given by an eigenvalue of the Laplacian on X_5. In general $z \neq 2$. We go on to show how such spaces arise as the near horizon limit of a branewave traveling along N D3 branes. We also show that despite the mixing of X_5 and AdS_5 coordinates, a scalar field propagating in the ten dimensional background is equivalent (in the $z = 2$ case) to a scalar field in the five dimensional space (1.2) but with a renormalised mass.

Towards the end of this paper we construct two types of explicit renormalisation group flow solutions that are flows away from an ultraviolet (UV) Schrödinger invariance. The first is a novel solution that appears to describe our field theories with a finite energy density. The second are nonrelativistic deformations of the known Coulomb branch solutions of $\mathcal{N} = 4$ super Yang-Mills theory. We conclude with comments on the dual field theory interpretation of our solutions, and a rather long to do list.

2. A family of IIB Schrödinger solutions

2.1 Solutions with no fluxes

Consider the following IIB ansatz, which has the full Schrödinger symmetry

$$\begin{aligned} ds^2_{\rm IIB} &= R^2 \left(r^2 \left(-2dx^+dx^- + d\boldsymbol{x}^2\right) + \frac{dr^2}{r^2} - f(X_5) r^4 (dx^+)^2 + ds^2_{X_5} \right) , \qquad (2.1) \\ F_5 &= 4R^4 (1 + \star) \mathrm{vol}(X_5) \, . \qquad (2.2) \end{aligned}$$

Here X_5 can be any five dimensional Einstein manifold, i.e. $R^{X_5}_{mn} = 4\, g^{X_5}_{mn}$. What we have done is to insert a function $f(X_5)$, depending on the coordinates on X_5, into the noncompact part of the metric. Thus the spacetime describes a wave propagating in AdS_5 with a profile along the internal X_5 directions. A similar ansatz was considered in [26]. In fact, section 2.2 of their paper describes a specific example of the solutions we shall discuss in the following subsection.

The wave in AdS_5 in (2.1) only alters the R_{++} component of the Ricci tensor relative to the pure AdS_5 background. It is easy to check that our ansatz solves the Einstein frame IIB equations of motion

$$R_{MN} = \frac{1}{96} F_{MABCD} F_N{}^{ABCD} \, , \qquad (2.3)$$

provided that

$$-\nabla^2_{X_5} f = 12 f \, . \qquad (2.4)$$

If f diverges at any point this will introduce singularities into the spacetime. These are not curvature singularities, but rather infinite tidal forces that appear for geodesic observers. For a recent discussion see for instance [27]. This is sufficient for time evolution in the spacetime to be ill-defined, so we should require that f is regular. Therefore (2.4) implies that f should be a scalar harmonic on X_5 with a specific eigenvalue.

When we consider general z below, we will find that the scalar harmonic eigenvalue depends on z. For the moment we can ask which Einstein manifolds have a scalar harmonic with eigenvalue 12, as required by (2.4). For S^5 we have the spectrum

$$-\nabla^2_{S^5} = \ell(\ell+4)\,. \tag{2.5}$$

So the $\ell = 2$ harmonics give solutions. The degeneracy of $\ell = 2$ on S^5 is 20, so $AdS_5 \times S^5$ has a twenty dimensional family of solutions of the form (2.1). The degeneracy can be obtained as the number of tracefree quadratic forms in six variables.

In fact, any Sasaki-Einstein manifold has eigenvalues of the form (2.5), see for instance [28]. However, in general ℓ does not run over all the positive integers, so we are not guaranteed the existence of a mode with eigenvalue 12. A familiar case where the spectrum is again known exactly is the homogeneous Sasaki-Einstein manifold $T^{1,1}$. The spectrum is (see e.g. [29])

$$-\nabla^2_{T^{1,1}} = 6\left(\ell_1(\ell_1+1) + \ell_2(\ell_2+1) - \frac{r^2}{8}\right)\,. \tag{2.6}$$

Here r is an integer, and ℓ_1, ℓ_2 can be integers or half integers, constrained as specified in for instance [29, 30]. There are precisely two combinations with the required eigenvalue: $(r, \ell_1, \ell_2) = (0, 1, 0)$ and $(0, 0, 1)$. Because ℓ_1 and ℓ_2 label standard spherical harmonics on S^2, this means that we have a total of six modes in this case, and hence a six parameter family of solutions.

It is interesting to note that if an Einstein manifold admits eigenvalues with $-\nabla^2_{X_5} = 12$, then the corresponding $AdS_5 \times X_5$ compactification has a scalar mode in AdS_5 which saturates the Breitenlohner-Freedman bound for stability. This is because certain metric and five form fluctuations about $AdS_5 \times X_5$ have masses $m^2R^2 = -\nabla^2_{X_5} + 16 \pm 8\sqrt{-\nabla^2_{X_5} + 4}$ (see e.g. [31]). Putting $-\nabla^2_{X_5} = 12$ leads to the Breitenlohner-Freedman bound for AdS_5: $m^2R^2 = -4$.

On any X_5, a nontrivial spherical harmonic has regions in which it is both positive and negative. This follows from orthogonality of the harmonic with a constant function. Therefore our function $f(X_5)$ appearing in the spacetime (2.1) will be negative in certain regions of the internal space. One might therefore worry about stability [27, 33] and geodesic completeness [32] of the spacetime. We will return to these points below. First we shall show how NSNS flux can be added to our solution (2.1) to allow an interpolation between these solutions without flux and the known solutions (1.5).

2.2 Solutions with fluxes

We can now add a NSNS two form potential to our solution (2.1). This will allow the family of solutions to connect onto the solutions (1.5). The ansatz is now

$$ds^2_{\rm IIB} = R^2\left(r^2\left(-2dx^+dx^- + d\boldsymbol{x}^2\right) + \frac{dr^2}{r^2} - f(X_5)r^4(dx^+)^2 + ds^2_{K-E} + \eta^2\right)\,, \tag{2.7}$$

$$B_2 = \sigma e^{\Phi/2}R^2r^2dx^+\wedge\eta\,, \tag{2.8}$$

$$F_5 = 4R^4(1+\star){\rm vol}(X_5)\,. \tag{2.9}$$

Here we again require X_5 to be Sasaki-Einstein and use the same notations as above.

It is easy to check that the five form and three form field equations are satisfied exactly as they were for the solution (1.5), the function $f(X_5)$ does not appear in these equations. The Einstein equation becomes

$$R_{MN} = \frac{1}{96} F_{MABCD} F_N{}^{ABCD} + \frac{e^{-\Phi}}{4} \left(H_{MAB} H_N{}^{AB} - \frac{1}{12} g_{MN} H_{ABC} H^{ABC} \right) , \tag{2.10}$$

where $H_3 = dB_2$ and Φ is the (constant) dilaton. The last term in this equation vanishes because H_3 is null in (2.7). The Einstein equation is solved provided that

$$-\nabla^2_{X_5} f = 12 f - 12\,\sigma^2 . \tag{2.11}$$

From this equation we obtain

$$f = \sigma^2 + \tilde{f} , \tag{2.12}$$

where $\tilde{f}$ is, as previously in (2.4), a scalar harmonic on X_5 with eigenvalue 12. We see that these solutions are precisely a linear superposition of our solution (2.1) without fluxes, and the known solution (1.5). By tuning σ and the magnitude of $\tilde{f}$ we interpolate between these two limits. Thus we obtain a 21 dimensional family of solutions from $AdS_5 \times S^5$ and a 7 dimensional family from $AdS_5 \times T^{1,1}$. We can also note that by taking σ to be sufficiently large relative to $\tilde{f}$, the function f appearing in the metric will become positive everywhere. This fact will shortly have consequences in our stability analysis.

3. Scalar field fluctuations

One might have expected that the mixing of internal and noncompact coordinates in our solutions (2.7) would complicate the study of fluctuations about the background. This is not the case. We will now show that, for scalar fields at least, the only effect of the mixing is to 'renormalise' the effective Kaluza-Klein mass of the modes.

The linearised equation of motion for a (massive) ten dimensional scalar field is

$$\nabla^2 \phi = m^2 \phi . \tag{3.1}$$

It is easily seen that we can solve equation (3.1) by writing

$$\phi = \Phi(r) Y_\lambda(X_5)\, e^{-i\omega x^+ + i\boldsymbol{k}\cdot\boldsymbol{x} - iMx^-} . \tag{3.2}$$

Here ω is the frequency, $\boldsymbol{k}$ the wavevector and M the rest mass. The eigenvalue λ is the solution to a Schrödinger-like equation on X_5

$$\left[-\nabla^2_{X_5} + M^2 f\right] Y_\lambda = \lambda\, Y_\lambda . \tag{3.3}$$

The function $\Phi(r)$ solves

$$\frac{d^2\Phi}{dr^2} + \frac{5}{r}\frac{d\Phi}{dr} + \frac{2M\omega - \boldsymbol{k}^2}{r^4}\Phi = \frac{m^2 + \lambda}{r^2}\Phi . \tag{3.4}$$

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Therefore we see that the only effect of the mixing of internal and noncompact coordinates is to 'renormalise' the effective mass of the five dimensional field: $m^2 \to m^2 + \lambda$. The wave equation separates even though the background is not a direct product. If the rest mass $M = 0$, then there is no dependence on f and Y_λ are simply the spherical harmonics on X_5. For nonzero M, the functions Y_λ solve a modified equation on X_5.

The five dimensional equation (3.4) is easily solved [14, 15] to give, up to normalisation, the modified Bessel function

$$\Phi = \frac{K_\nu(p/r)}{r^2}\,, \tag{3.5}$$

where

$$p = \sqrt{\boldsymbol{k}^2 - 2M\omega}\,, \qquad \nu = \sqrt{m^2 + \lambda + 4}\,. \tag{3.6}$$

We have imposed regularity of the solution at the horizon $r = 0$ (assuming that p is real). The different modes of this equation correspond to scalar operators $\mathcal{O}$ with scaling dimension

$$\Delta_{\mathcal{O}} = 2 + \nu\,. \tag{3.7}$$

If $0 < \nu < 1$ we can also take $\Delta_{\mathcal{O}} = 2 - \nu$ [14]. Thus we see explicitly that the effect of our function f is to shift the scaling dimension of the operators dual to bulk scalar fields.

4. Stability and geodesic completeness: a phase transition

We noted that in our solution without flux (2.1), the function f appearing in the metric would necessarily be negative over some region of X_5. This is potentially associated with various spacetime pathologies, see e.g. [27, 33], two of which we shall now investigate. On the other hand, in the previously known background (1.5), f is positive everywhere. In our family of solutions (2.7) interpolating between these limits, f becomes negative on some critical hypersurface in the space of solutions. A pathology arising on this surface would indicate an interesting zero temperature phase transition in the dual family of nonrelativistic field theories. We will see that while the space remains geodesically complete across the transition to nonpositive f, there is indeed an instability once f becomes sufficiently negative.

4.1 Appearance of an unstable scalar fluctuation

The first question we consider is stability against scalar field perturbations, such as the dilaton. Stability in pp wave backgrounds is complicated by various factors such as the lack of global hyperbolicity [34]. See however the comments in, for instance, [33]. We shall take the following simple approach that seems appropriate for the present context.

Firstly, we fix the sign of the rest mass $M \geq 0$. This is a choice of direction of the Killing vector generating translations in x^-. Next, perform a simple change of variables on the effective five dimensional equation for a scalar field in (3.4) above. The objective is to put the equation in Schrödinger form. Let

$$u = \frac{1}{r}\,, \qquad \Phi = u^{3/2}\Psi\,. \tag{4.1}$$

Furthermore, we will put $\boldsymbol{k} = 0$, as zero momentum gives the potentially most unstable mode. Then from (3.4) we obtain

$$-\frac{d^2\Psi}{du^2} + \left(\nu^2 - \frac{1}{4}\right)\frac{1}{u^2}\Psi = 2M\omega\Psi\,. \tag{4.2}$$

Recall that ν is given by (3.6) above. This is a very well studied Schrödinger equation (with 'Energy' $E = 2M\omega$). In particular, it is easy to see that when $\nu^2 < 0$ there is a continuum of bound states with arbitrarily low energies. This means that there are normalisable solutions to (4.2) with arbitrarily negative $M\omega$.

An instability is usually signalled in general relativity by an exponentially growing mode, in which ω is imaginary. This is not what we have here. In the usual case of pure AdS space, $f = 0$, the solutions we have found for $\nu^2 < 0$ correspond to modes with masses below the Breitenlohner-Freedman bound [35], that are growing exponentially in the boundary Minkowski time $t = x^+ + x^-$. In the present setup, we have periodically identified x^-, and hence the rest mass M must be real. This constraint, and self-adjointness of the Schrödinger equation, requires ω to be real also and thus there is no exponential growth.

However, given that we have taken $M > 0$, then $M\omega$ negative requires ω negative. We will see shortly that large M is necessary to obtain $\nu^2 < 0$. Nonetheless, we can take M large but finite. The fact that $M\omega$ is unbounded below then implies that we have normalisable solutions with ω arbitrarily negative. The frequency ω is nothing but the energy of the dual field theory, with respect to the nonrelativistic time x^+. Our analysis will therefore imply that the Hamiltonian of the dual nonrelativistic theory becomes unbounded below when $\nu^2 < 0$. This is indeed an instability. We do not see exponential growth because we have taken a free scalar field. Generic interactions will send the system into energetic free-fall.

We can also note from (3.7) above that $\nu^2 < 0$ corresponds to an imaginary scaling dimension for the dual operator $\mathcal{O}$. This reinforces the idea that the system is unstable in these cases.

The question is therefore: when do modes appear with $\nu^2 < 0$? Recall that $\nu^2 = m^2 + \lambda + 4$. The standard $AdS_5 \times X_5$ vacua of IIB string theory are stable provided that there are no five dimensional modes with masses below the Breitenlohner-Freedman bound: $m^2 + 4 \geq 0$. Let us assume we are compactifying on such an X_5. This class of five manifolds includes all Sasaki-Einstein manifolds. Our solutions will therefore become unstable if λ can become sufficiently negative. Recall that the allowed values of λ are eigenvalues of the operator (3.3) on X_5. It is clear that if f is positive in (3.3) then λ must also be positive. Thus

$$f > 0 \quad \Rightarrow \quad \text{Stability.} \tag{4.3}$$

Of course we have only shown stability against the particular mode (3.1).

We will now show that if we are sufficiently close in parameter space to the solution (2.1), with no three form flux, then by taking M sufficiently large an instability will appear. We will not show this in generality, but rather for the case of S^5. Let us scale M to be parametrically large, which we can do, as the theory contains excitations with arbitrarily large particle number. It then follows from (3.3) that a sufficient condition for an arbitrarily negative λ is if we can find a test function Y on S^5 such that the potential energy is negative. That is

$$\exists Y \quad \text{s.t.} \quad \int_{S^5} f Y^2 d\Omega < 0 \quad \Rightarrow \quad \text{Instability.} \tag{4.4}$$

The potential term will eventually dominate over the kinetic term in (3.3) if we take M to be sufficiently large.

If we introduce cartesian coordinates $\{x^i\}$ on $\mathbb{R}^6$, then we have from above that

$$f = \sigma^2 + a_{ij} x^i x^j \,, \tag{4.5}$$

with a_{ij} a symmetric tracefree real matrix. The function f is of course restricted to the unit $S^5 \subset \mathbb{R}^6$. Let us now take as a test function an $\ell = 1$ harmonic on the five sphere. That is: $Y = b_k x^k$. This gives

$$\begin{aligned} \int_{S^5} f Y^2 d\Omega &= \sigma^2 b_k b_l \langle x^k x^l \rangle_{S^5} + a_{ij} b_k b_l \langle x^i x^j x^k x^l \rangle_{S^5} \\ &= \mathrm{Vol}(S^5) \left(\frac{\sigma^2 b_i b_i}{6} + \frac{b_i a_{ij} b_j}{24} \right) . \end{aligned} \tag{4.6}$$

Denote the eigenvalues of a_{ij} by $a^{(i)}$. A tracefree symmetric matrix necessarily has at least one negative eigenvalue. Therefore

$$4\,\sigma^2 < |\min_i a^{(i)}| \quad \Rightarrow \quad \text{Instability.} \tag{4.7}$$

This relationship shows that if σ (i.e. the three form flux) is sufficiently small, the theory becomes unstable, as advertised. It may well be that the theory becomes unstable at a larger value of σ than that given in (4.7). The results (4.3) and (4.7) leave the stability in the range $\sigma^2 \in \left[\frac{1}{4}|\min_i a^{(i)}|, |\min_i a^{(i)}|\right]$ undetermined.

To summarise the results from this section: Equations (4.3) and (4.7) give conditions on the function appearing in (4.5) to give stable and unstable spacetimes, respectively. We have not found the precise hypersurface in $\{\sigma^2, a_{ij}\}$ space where an instability sets in, but we have bounded it on both sides.

It is worth highlighting that this instability is catastrophic for the spacetime. As soon as the the criterion $\sigma \leq \sigma_{\text{crit.}}(a^{(i)})$ is satisfied then all modes in the theory with sufficiently large particle number become unstable at once. It would be extremely interesting to understand the field theoretic dual description of this process. Regions in which f is negative have a different causal structure to those with positive f. The positive f regions are causally non-distinguishing, as one expects for a dual to a non-relativistic field theory. Perhaps the instability removes the negative f regions from the spacetime?

4.2 Geodesic completeness

Another potential pathology of the spacetime that can arise when f is not positive is geodesic incompleteness. In particular, we have to check that no timelike or null geodesics

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can reach infinity in a finite affinely parametrised worldline time [32]. For simplicity let us consider a geodesic that is at a constant location on X_5. This is possible at the stationary points of $f(X_5)$. It is easy to check that relaxing this restriction will not alter our conclusions — the geodesic would simply oscillate in the X_5 directions as it moves out towards infinity.

It is straightforward to check that to leading order at large r, (timelike or null) geodesics satisfy

$$\left(\frac{dr}{ds}\right)^2 \sim -(M^2 + fE^2)r^2\,, \tag{4.8}$$

where s is an affine parameter and M^2 and E^2 are positive constants (M can be zero). If f is positive everywhere, then this equation is inconsistent and hence geodesics cannot reach infinity at all. If f is negative then, by taking E sufficiently large, the right hand side of (4.8) can become positive. The equation is easily solved to give

$$r \sim e^{ms}\,, \tag{4.9}$$

for some positive constant m. Therefore we see that it takes an infinite affine time for geodesics to reach infinity. The spacetime is geodesically complete in all cases. This result is closely related to the regularity of plane wave spacetimes [36]. We shall see the connection to plane waves explicitly in section 6 below.

5. Many solutions with $z \neq 2$

Generalising our above construction to a large number of values of z is straightforward and leads to an interesting observation. Consider the ansatz

$$ds^2_{\text{IIB}} = R^2\left(r^2\left(-2dx^+dx^- + d\boldsymbol{x}^2\right) + \frac{dr^2}{r^2} - f(X_5)r^{2z}(dx^+)^2 + ds^2_{K-E} + \eta^2\right)\,, \tag{5.1}$$

$$F_5 = 4R^4(1+\star)\text{vol}(X_5)\,. \tag{5.2}$$

Note that we have not included a B_2 field. The B_2 equation is not easy to satisfy for general z, which is one of the reasons why the papers [19−21] concentrated on the case $z = 2$. With the above ansatz (5.1) the Einstein equation is satisfied provided that

$$-\nabla^2_{X^5} f = 4(z^2-1)f\,. \tag{5.3}$$

Once again, we have regular solutions whenever the manifold X_5 admits a scalar harmonic with a specific eigenvalue.

For the $AdS_5 \times S^5$ case, equation (5.3), together with (2.5), implies that we have solutions for all

$$z = \frac{\ell+2}{2}\,. \tag{5.4}$$

That is to say, z can take an integer or half-integer value greater than one. The dimensionality of the family of solutions obtained is given by the number of tracefree rank ℓ symmetric tensors on $\mathbb{R}^6$:

$$\#(z) = \frac{1}{12}(1+\ell)(2+\ell)^2(3+\ell) = \frac{z^2(4z^2-1)}{3}\,. \tag{5.5}$$

Clearly this becomes very large for large z.

For a general X_5, the allowed values of z will be a sequence of irrational numbers, directly related to the scalar eigenvalues of X_5 through (5.3). This gives a rather intriguing connection between spectral properties of Einstein manifolds and dynamical critical exponents. A general statement that can be made is the following. The Lichnerowicz theorem, see e.g. [37], applied to five dimensions, says that the lowest nontrivial Laplacian eigenvalue of a five dimensional Einstein manifold with $R_{mn}^{X_5} = 4g_{\mathrm{mn}}^{X_5}$ is bounded below

$$-\nabla^2_{X_5} \geq 5\,, \tag{5.6}$$

with equality only for the case of S^5. Using our relation (5.3) we obtain

$$z \geq \frac{3}{2}\,. \tag{5.7}$$

Of course there is also the $z = 1$ possibility of pure AdS which corresponds to the trivial (zero) eigenvalue. Thus our construction will only enable us to obtain theories with dynamical critical exponents satisfying (5.7).

We should note the following two facts. Firstly, for $z > 2$ the spacetime (5.1) is not geodesically complete. It is easy to see that at large r, and in regions of X_5 where f is negative, there are (null and timelike) geodesics that behave like

$$r \sim \frac{1}{(s-s_0)^{1/(z-2)}}\,. \tag{5.8}$$

Again, s is an affine parameter and s_0 a constant. These geodesics reach infinity at a finite affine parameter $s = s_0$. The geodesic incompleteness may possibly be related to the lack of scaling near infinity for $z > 2$ that was noted in [15]. It is not clear at this stage how important this result is given that the large radius region is problematic in any case [20]. This is because, even with a putative solution at finite number density, at large radius the periodic identification of the x^- circle means that wound strings become arbitrarily light and the supergravity description breaks down. In our solution of course, without number density, the circle is null everywhere.

Secondly, if $z < 2$ the spacetime is singular at the origin $r = 0$ (unless $z = 1$). Although the curvatures are finite, there are pp singularities in regions where f is negative. For instance, a radially infalling null geodesic experiences infinite tidal forces as $r \to 0$ in a parallelly propagating orthonormal frame. These are mild null singularities that may well disappear in a finite temperature solution.

We will not study these solutions further here. It would be interesting to investigate their stability. Our experience in the $z = 2$ case above and the absence of flux in these solutions might suggest that they are unstable. However, unlike we found in section 3 above, the ten dimensional wave equation in the background (5.1) does not separate nicely for $z \neq 2$, which complicates the analysis.

6. Undoing the near horizon limit: branewaves

It is instructive to write down a IIB geometry that asymptotes to a pp wave in flat space and which develops a Schrödinger symmetry in a near horizon limit. The solution is

$$
\begin{aligned}
ds^2_{\rm IIB} &= \frac{1}{H(r)^{1/2}}\left(-2dx^+dx^- + d\boldsymbol{x}^2 - f(X_5)r^2(dx^+)^2\right) + H(r)^{1/2}\left(dr^2 + r^2 ds^2_{X^5}\right)\,, && (6.1)\\
ds^2_{X^5} &= ds^2_{K-E} + \eta^2\,, && (6.2)\\
B_2 &= \sigma e^{\Phi/2} r^2 dx^+ \wedge \eta\,, && (6.3)\\
F_5 &= \frac{-H'(r)}{H(r)^2}(1+\star)dr\wedge dx^+\wedge dx^-\wedge dx^1\wedge dx^2\,. && (6.4)
\end{aligned}
$$

Here $H(r)$ is the usual extremal D3 brane function

$$H(r) = 1 + \frac{R^4}{r^4}\,. \tag{6.5}$$

It is straightforward to check that the equation for the three form field strength is independent of the function H, and therefore remains satisfied for this more general metric ansatz. The Einstein equations (2.10) are satisfied provided that, once again,

$$-\nabla^2_{X_5} f = 12 f - 12\,\sigma^2\,. \tag{6.6}$$

As above, this is solved by letting $f = \sigma^2 + \tilde{f}$, with $\tilde{f}$ being a scalar harmonic on X_5 with eigenvalue 12. The near horizon limit is now $r \ll R$, which leads to the solution we found in previous sections. Although natural, this is not the unique near horizon limit. A different scaling limit would recover the usual $AdS_5 \times X^5$ geometry.

The solution (6.1) has an immediate interpretation as a D3 brane in a pp wave background. In fact, in [38] a class of solutions were found that includes (6.1) in the case with no B_2. Similar solutions were discussed in [39], although these had a wave profile along the D3 brane directions. In the case we are studying, the pp wave is travelling along the D3 brane, but has a profile orthogonal to the D3 brane. The connection to D3 branes in a pp wave background can be made especially explicit in the case of $X_5 = S^5$, as the geometry can be rewritten as

$$ds^2_{\rm IIB} = \frac{1}{H(r)^{1/2}}\left(-2dx^+dx^- + d\boldsymbol{x}^2 - [\sigma^2\delta_{ij} + a_{ij}]y^i y^j (dx^+)^2\right) + H(r)^{1/2} d\boldsymbol{y}^2\,. \tag{6.7}$$

Here $\boldsymbol{y}$ are the six coordinates transverse to the D3 brane worldvolume, $r^2 = \boldsymbol{y}^2$, and a_{ij} is a tracefree matrix of real numbers. We used the fact that the $\ell = 2$ harmonics on S^5 are inherited from quadratic polynomials in $\mathbb{R}^6$. This spacetime asymptotes to a homogeneous eight dimensional plane wave times $\mathbb{R}^2$. The near horizon Schrödinger group can be thought of as a remnant of the large symmetry of the plane wave. The tracefree part of the waveform originates from a purely metric ten dimensional solution, whereas the trace is supported by the null three form flux.

6.1 Supersymmetries

Given that the $D3$ brane and the homogeneous pp wave (without flux, $B_2 = 0$) both preserve half of the supersymmetries of IIB supergravity, it is natural to expect that this intersection of the two solutions will preserve one quarter.

It is straightforward to check that our solution (2.1) with $B_2 = 0$ indeed preserves a quarter of the supersymmetries. In the absence of NSNS three form flux, all that is needed is that the gravitino variation vanish

$$\delta\psi_M = \left(D_M + \frac{i}{192}\Gamma^{NPQRS}\Gamma_M F_{NPQRS} \right) \epsilon = 0\,. \tag{6.8}$$

We can introduce the tangent space gamma matrices through

$$\gamma^\alpha = e^\alpha{}_M \Gamma^M\,, \tag{6.9}$$

where $e^\alpha{}_M$ is the vielbein. The spinorial covariant derivative is then

$$D_M = \partial_M + \tfrac{1}{2}\omega_M^{\alpha\beta}\gamma_{\alpha\beta}\,. \tag{6.10}$$

We now proceed to separate the spinor ϵ into various components using projection operators. Firstly one can impose the chirality condition

$$\gamma_{11}\epsilon = -\epsilon\,, \tag{6.11}$$

where as usual $\gamma_{11} = -\gamma^{+-12r\cdots}$. Then we split

$$\epsilon = \epsilon_+ + \epsilon_- \equiv \tfrac{1}{2}(1 + i\gamma^{+-12})\epsilon + \tfrac{1}{2}(1 - i\gamma^{+-12})\epsilon\,, \tag{6.12}$$

and furthermore split

$$\epsilon_\pm = \epsilon_\pm^{(+)} + \epsilon_\pm^{(-)} \equiv -\tfrac{1}{2}\gamma^-\gamma^+\epsilon_\pm - \tfrac{1}{2}\gamma^+\gamma^-\epsilon_\pm\,. \tag{6.13}$$

Using this decomposition it is straightforward to solve the Killing spinor equations (6.8) for the near horizon Schrödinger geometry (2.1). The result is

$$\epsilon_\pm^{(+)} = 0\,, \quad \epsilon_-^{(-)} = 0\,, \quad \epsilon_+^{(-)} = \sqrt{r}\, h(X_5)\,\eta_+^{(-)}\,, \tag{6.14}$$

where $\eta_+^{(-)}$ is a constant spinor consistent with the projections above and $h(X_5)$ is an operator that projects onto Killing spinors on the internal space X_5.

Introducing the wave therefore breaks any pre-existing symmetries down to one quarter. In particular, if $X_5 = S^5$ the background (2.1) is 1/4 BPS, while if $X_5 = T^{1,1}$ or any other Sasaki-Einstein space, the background is 1/16 BPS. The surviving symmetries are half of the supertranslations. This suggests that the supersymmetry algebra of this background may be the algebra with eight supercharges found in [40]. Other recent interesting discussions of a super Schrödinger symmetry for AdS/CFT include [41, 42]. A systematic approach to Schrödinger superalgebras was presented in [43].

Including a nonzero B_2 is likely to break the supersymmetry altogether, as noted in [19, 20].

7. A renormalisation group flow

In this section we study a renormalisation group flow. An interesting novelty compared with the well studied Poincaré invariant renormalisation group flows in AdS/CFT is that here the time and spatial parts of the metric can scale differently as a function of the radial distance (this fact has been used in [44] and [45] to find flows that interpolate between fixed points with different values of the speed of light and z, respectively). This allows more freedom in finding solutions.

The following relatively simple ansatz is consistent:

$$\begin{aligned} ds^2_{\rm IIB} &= R^2\left(-2s(r)dx^+dx^- + t(r)d\boldsymbol{x}^2 + \frac{dr^2}{r^2} - f(X_5)m(r)(dx^+)^2 + p(r)ds^2_{X_5}\right), && (7.1)\\ F_5 &= 4R^4(1+\star)\mathrm{vol}(X_5)\,. && (7.2)\end{aligned}$$

As before $-\nabla^2_{X_5} f = 12f$. The Einstein equations become a set of nonlinear equations for the four functions $\{s,t,m,p\}$. There is a first order constraint

$$r^2\left(\frac{1}{4}\frac{s'^2}{s^2} + \frac{1}{4}\frac{t'^2}{t^2} + \frac{5}{2}\frac{p's'}{ps} + \frac{5}{2}\frac{p't'}{pt} + \frac{5}{2}\frac{p'^2}{p^2} + \frac{t's'}{ts}\right) - \frac{10}{p} + \frac{4}{p^5} = 0\,, \qquad (7.3)$$

and three second order equations

$$\begin{aligned} m'' &= \left(\frac{s'}{s} - \frac{t'}{t} - \frac{5}{2}\frac{p'}{p} - \frac{1}{r}\right)m' + \left(-\frac{s'^2}{s^2} + \frac{12}{r^2p} + \frac{8}{r^2p^5}\right)m\,, && (7.4)\\ s'' &= \left(-\frac{t'}{t} - \frac{5}{2}\frac{p'}{p} - \frac{1}{r}\right)s' + \frac{8s}{r^2p^5}\,, && (7.5)\\ t'' &= \left(-\frac{s'}{s} - \frac{5}{2}\frac{p'}{p} - \frac{1}{r}\right)t' + \frac{8t}{r^2p^5}\,. && (7.6)\end{aligned}$$

From these equations and the first order equation (7.3) it is straightforward to obtain an equation for p''. Note the structure of the equations: there are three equations for $\{s,t,p\}$ and then a decoupled equation for m. Therefore any solution will be a (generally non-Poincaré invariant) renormalisation group flow of the $\mathcal{N}=4$ theory that is then deformed into a nonrelativistic background, without backreaction on the initial metric functions.

Remarkably, it is possible to find an explicit analytic solution to these equations. In fact, one can find the general solution with a constant size S^5, that is $p=1$. The remaining radial functions become

$$\begin{aligned} m(r) &= \left(\frac{r^4-r_0^4}{r^4+r_0^4}\right)^{\sqrt{3}/2}\frac{\sqrt{r^8-r_0^8}}{r_0^8}\,E\left(1-\frac{r_0^8}{r^8}\right), && (7.7)\\ s(r) &= \left(\frac{r^4-r_0^4}{r^4+r_0^4}\right)^{\sqrt{3}/2}\frac{\sqrt{r^8-r_0^8}}{r^2}\,, && (7.8)\\ t(r) &= \left(\frac{r^4+r_0^4}{r^4-r_0^4}\right)^{\sqrt{3}/2}\frac{\sqrt{r^8-r_0^8}}{r^2}\,. && (7.9)\end{aligned}$$

Here $E(k)$ is a complete elliptic integral of the second kind.[1] Note that $E(0) = \pi/2$. There is a second independent solution which has a logarithmic divergence as $r \to r_0$, which we have not written here.

The first fact we can note is that this metric asymptotes as $r \to \infty$ to our original solution (2.1). Thus, we asymptotically develop a Schrödinger symmetry. On the other hand, the solution is singular at $r = r_0$. It is easy to check that the Riemann tensor squared diverges as $r \to r_0$. However, it is a null singularity (this is seen from the fact that it takes an infinite boundary time for an infalling null observer to reach the singularity) and therefore likely to be admissible as a solution in the full string theory [46].

If we had taken $m = 0$, this flow would have a simple interpretation. The functions s and t appearing in the metric (7.1) are dual to components of the energy momentum tensor in the (3+1 dimensional) $\mathcal{N} = 4$ theory. There is a vacuum expectation value but no source for these modes. Therefore we would be looking at the theory in a state in which the expectation value for the pressure is anisotropic. This is not especially natural in the absence of anisotropic sources. However, once we have $m \neq 0$ there is a different interpretation.

Viewed as a dual to a nonrelativistic theory, we identify the x^- coordinate which is now no longer a spacetime coordinate in the dual (2+1 dimensional) field theory. According to [14, 21], the r^2 coefficient of $(dx^+)^2$ couples to the particle number density of the dual theory while in our solution the subleading term in $m(r)$ is a constant. Therefore it seems that this solution corresponds to a finite energy density but zero particle density. This statement is furthermore supported by the absence of a $(dx^-)^2$ term, which appeared in the finite particle density solutions of [19–21].

Another consistent simplification of the equations above is to set $s = t$. We have not been able to solve the resulting equations analytically. This flow, with $m = 0$, would be a Poincaré invariant state of the $\mathcal{N} = 4$ theory and therefore less specific to the nonrelativistic setup.

8. Coulomb branch solutions

The methods we have been using are easily adapted to produce Coulomb branch solutions that asymptote to nonrelativistic backgrounds. Our starting point will be the explicit supersymmetric and asymptotically AdS backgrounds that were constructed in [47]. There are five such backgrounds, which preserve an $\mathrm{SO}(n) \times \mathrm{SO}(6-n)$ subgroup of the SO(6) symmetry of the $\mathcal{N} = 4$ theory. The fact that these backgrounds only involve the metric and five form field strength make them especially amenable to non-relativistic generalisation. Some solutions related to those we shall present shortly, but with different asymptotics, were found in [48].

[1] Recall that $E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta$. The elliptic integral may also be expressed in terms of hypergeometric functions or associated Legendre functions. There seems to be a bug in (some versions at least of) Maple in simplifying these elliptic integrals.

8.1 Asymptotically Schrödinger solution with SO(2) × SO(4) symmetry

We will write down directly the Coulomb branch solution with the nonrelativistic deformation.[2] The metric and five form of our solution are

$$\begin{aligned} ds^2 &= \zeta r^2\left(-2dx^+dx^- + d\boldsymbol{x}^2 + \frac{dr^2}{r^4\lambda^6} - m(r,S^5)(dx^+)^2\right) \\ &\qquad\qquad + \zeta d\theta^2 + \frac{\cos^2\theta}{\zeta}d\Omega^2_{S^3} + \frac{\lambda^6\sin^2\theta}{\zeta}d\Omega^2_{S^1}\,, \\ F_5 &= \left[r\left(L^2+4r^2+L^2\cos 2\theta\right)dr - L^2r^2\sin 2\theta d\theta\right]\wedge dx^+\wedge dx^-\wedge dx\wedge dy\,, \end{aligned} \tag{8.1}$$

with the remaining components of F_5 obtained by Hodge dualising. We introduced the functions

$$\lambda^6 = 1+\frac{L^2}{r^2}\,, \qquad \zeta^2 = 1+\frac{L^2}{r^2}\cos^2\theta\,. \tag{8.2}$$

Here L gives a notion of the distance between the smeared $D3$ branes. At $L=0$ the solution reduces to a gravitational wave in pure AdS. Setting $m=0$ gives the solution in [47]. Introducing a nonzero m will still solve the IIB equations of motion (2.3) provided that m is harmonic on the spacetime with $m=0$, i.e. $\nabla^2 m = 0$. Writing this out explicitly gives

$$\frac{1}{r^3\zeta}\frac{d}{dr}\left(\lambda^6 r^5\frac{dm}{dr}\right)+\frac{1}{\zeta}\frac{1}{\cos^3\theta\sin\theta}\frac{d}{d\theta}\left(\cos^3\theta\sin\theta\frac{dm}{d\theta}\right)+\frac{\zeta}{\cos^2\theta}\nabla^2_{S^3}m+\frac{\zeta}{\lambda^6\sin^2\theta}\nabla^2_{S^1}m=0\,. \tag{8.3}$$

The differential equation (8.3) can be separated easily if we take m to be independent of the coordinates of the S^3 and S^1 in the spacetime (8.1). To find a regular solution on the S^5 we must therefore find spherical harmonics on S^5 that only depend on the coordinate θ. This is the coordinate that arises if we parametrise $S^5\subset\mathbb{R}^6$ by

$$\vec{x} = \cos\theta\,\vec{n}_4 + \sin\theta\,\vec{n}_2\,, \tag{8.4}$$

where $\vec{n}_4$ and $\vec{n}_2$ are unit vectors taking values in an orthogonal $\mathbb{R}^4$ and $\mathbb{R}^2$, respectively, in $\mathbb{R}^6$. Recall that spherical harmonics on S^5 with eigenvalue $\ell(\ell+4)$ are given by tracefree symmetric tensors of rank ℓ on $\mathbb{R}^6$. Requiring SO(4) × SO(2) invariance restricts us to tensors of the form

$$c_{a_1\cdots a_{2M}}x^{a_1}\cdots x^{a_{2M}} = \sum_{N=0}^{M}\alpha_N(x_1^2+x_2^2+x_3^2+x_4^2)^N(x_5^2+x_6^2)^{M-N}\,. \tag{8.5}$$

The tracefree condition (after allowing for the symmetry in this tensor) then imposes M relations between the $M+1$ different α_N. Thus we can find a spherical harmonic solution for any even $\ell = 2M$. By our formula (5.4) we thus obtain Coulomb branch solutions that asymptote to nonrelativistic backgrounds for any positive integer $z\in\mathbb{Z}^+$. Let us write explicitly the solution in the case $\ell=2$. Using the parametrisation (8.4) and furthermore solving the remaining radial equation in (8.3) leads to the solution

$$m_{(z=2)} = a\left(1+\frac{3r^2}{2L^2}\right)\left(-\frac{2}{3}+\cos^2\theta\right)\,. \tag{8.6}$$

[2] In this section we set the AdS radius $R=1$ for simplicity.

Here a is a free parameter. There may of course be other more complicated solutions to (8.3) in which the variables do not separate. It is pleasing that the simplest regular solution turns out to have the properties we are interested in. The solution is a gravitational wave in a supersymmetric Coulomb branch geometry that asymptotes to our nonrelativistic dual spacetime (2.1), with a particular choice for the $\ell = 2$ spherical harmonic in that solution.

8.2 SO(3) × SO(3) and SO(5) symmetric solutions

The treatment of the other two possible symmetries is entirely analogous, and so we shall be brief. We take the SO(3) × SO(3) case first. The solution is

$$\begin{aligned} ds^2 &= \zeta r^2 \lambda \left(-2dx^+ dx^- + d\boldsymbol{x}^2 + \frac{dr^2}{r^4\lambda^6} - m(r, S^5)(dx^+)^2 \right) \\ &\qquad + \frac{\zeta}{\lambda} d\theta^2 + \frac{\cos^2\theta}{\zeta\lambda} d\Omega^2_{S^2} + \frac{\lambda^3 \sin^2\theta}{\zeta} d\Omega^2_{S^2} \,, \end{aligned} \tag{8.7}$$

$$\begin{aligned} F_5 &= \Bigg[\frac{L^4 + 8L^2r^2 + 8r^4 + L^2(L^2 + 2r^2)\cos 2\theta}{2\sqrt{L^2 + r^2}} dr \\ &\qquad - L^2 r \sqrt{L^2 + r^2} \sin 2\theta d\theta \Bigg] \wedge dx^+ \wedge dx^- \wedge dx \wedge dy \,, \end{aligned} \tag{8.8}$$

with the remaining components of F_5 obtained by Hodge dualising. While ζ is as before in (8.2), we now have

$$\lambda^4 = 1 + \frac{L^2}{r^2} \,. \tag{8.9}$$

As before we can find solutions with integer values of z. The solution that has an asymptotic Schrödinger symmetry is

$$m_{(z=2)} = a \frac{L^4 + 8L^2r^2 + 8r^4}{L^2 r \sqrt{L^2 + r^2}} \left(-\frac{1}{2} + \cos^2\theta \right) \,, \tag{8.10}$$

where again a is a free parameter.

For the case with SO(5) symmetry, the solutions take the form

$$\begin{aligned} ds^2 &= \frac{\zeta r^2}{\lambda^3} \left(-2dx^+ dx^- + d\boldsymbol{x}^2 + \frac{dr^2}{r^4\lambda^6} - m(r, S^5)(dx^+)^2 \right) + \zeta\lambda^3 d\theta^2 + \frac{\lambda^3\cos^2\theta}{\zeta} d\Omega^2_{S^5} \,, \\ F_5 &= \Bigg[\frac{r^2(3L^4 + 12L^2r^2 + 8r^4 + L^2(3L^2 + 2r^2)\cos 2\theta)}{2(L^2 + r^2)^{3/2}} dr \\ &\qquad - \frac{L^2 r^3}{\sqrt{L^2 + r^2}} \sin 2\theta d\theta \Bigg] \wedge dx^+ \wedge dx^- \wedge dx \wedge dy \,, \end{aligned} \tag{8.11}$$

with the remaining components of F_5 obtained by Hodge dualising. For this solution we have

$$\lambda^{12} = 1 + \frac{L^2}{r^2} \,. \tag{8.12}$$

The solution with an asymptotic Schrödinger symmetry has

$$m_{(z=2)} = \left(a \left(\frac{5}{6} + \frac{r^2}{L^2} \right) + b \frac{\sqrt{r^2 + L^2}\left(24r^4 + 8L^2r^2 - L^4\right)}{L^2 r^3} \right) \left(-\frac{5}{6} + \cos^2\theta \right) \,. \tag{8.13}$$

Unlike in the other cases, there are two parameters here, a and b. Both of these solutions have an asymptotic Schrödinger symmetry. One of them has a divergence as $r \to 0$, but given that the background itself is singular in this limit it is not clear that this is sufficient a pathology to throw away the solution. As before, there are also solutions for all other positive integer z.

There were two further metrics considered in [47]. These are related to the SO(2) × SO(4) and SO(5) metrics presented above by letting $L^2 \to -L^2$. The metric with SO(5) symmetry and the sign of L^2 inverted is believed to be nonphysical.

We found above that introducing our pp wave into $AdS_5 \times S^5$ broke all the supersymmetries except for half of the supertranslations. The Coulomb branch solutions for $\mathcal{N} = 4$ super Yang Mills theory break the superconformal symmetries but preserve all the supertranslations [47]. Therefore, it seems likely that the solutions we have found in this section will preserve half of the supertranslation symmetries.

9. Discussion

In this paper we have presented several solutions to IIB supergravity with features that seem likely to be of interest as duals for nonrelativistic quantum critical theories in 2+1 dimensions. We initiated a study of basic features of these solutions including stability, regularity and renormalisation flow generalisations. In this discussion we note important open questions and comment on the dual interpretation of these backgrounds.

Given that our backgrounds have (at least) a manifest 2+1 dimensional Galilean symmetry, it seems clear that they are dual to nonrelativistic theories defined on a 2+1 dimensional spacetime. In particular, the mixing between the 'AdS' and internal coordinates in the bulk should not alter this fact. Furthermore, our $z = 2$ solutions are continuously connected to the embedding found in [19 − 21]. It therefore seems likely that, as with those solutions, our bakgrounds are dual to (differing) twisted DLCQs of $\mathcal{N} = 4$ Super Yang-Mills theory. The dual description of the solutions with $z \neq 2$ is possibly less clear from this point of view. However, we noted that they share with the $z = 2$ cases the interpretation as the near horizon limit of a branewave.

It is of interest for several reasons to generalise the solutions we have found to a finite temperature and also a finite number density, with a $(dx^-)^2$ term in the metric. One reason is that most potential applications to experimental systems will be at least at finite number density and often at finite temperature. Another reason is that, as we have mentioned several times above, in solutions with vanishing temperature and number density the periodically identified circle (1.3) is null. This is expected to invalidate the supergravity approximation as wrapped strings can become light [20].

The mixing in our solution (2.1) between the noncompact and the X_5 directions makes it hard to find generalisations with a $(dx^-)^2$ component in the metric. This is what we would need to make the identified circle spacelike. In fact, all of the solutions discussed in this paper seem to be an application of the Garfinkle-Vachaspati solution generating technique [49, 50] (for a recent review of this method in a string theory context see for instance [38]). This method requires a null (hypersurface-orthogonal) Killing vector field, which will not be present in finite temperature solutions, or more generally if ∂_{x^-} is spacelike. A different approach will be required to find the finite temperature solutions.

Restricting to the solutions we have found in this paper, there are several immediate remaining questions. We did not analyze the stability of the solutions with $z \neq 2$ and we did not consider the stability of any of the backgrounds against other modes such as gravitational perturbations. We also did not find the explicit form of the unstable scalar mode.

It seems clear that our constructions can straightforwardly be generalised to different spacetime dimensions. It would also be interesting to generalise the renormalisation group flow solution of section 7 to include a B_2 field, and also to different values of z.

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